Introduction to SET Theory

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Cards

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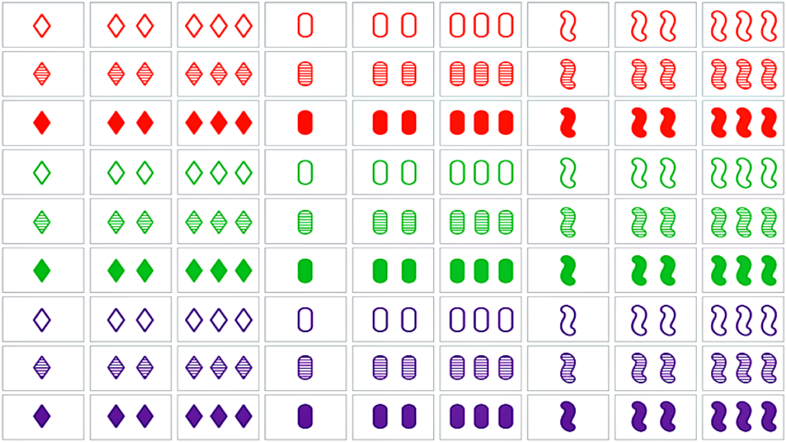

What is a SET?

An example of a SET.

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A **non-example** of a SET.

Encoding SET as a Vector Space

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Since each card is uniquely determined by its 4 attributes, each of which have 3 options, we can encode cards as elements of $\mathbb{F}_3^4$ (where $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ denotes the field with 3 elements).

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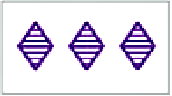
For example:

 $\longleftrightarrow$ Three Striped Purple Diamonds $\longleftrightarrow$ $(3, 2, 3, 1)$

SETs in $\mathbb{F}_3^4$

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Question: What does a SET correspond to in $\mathbb{F}_3^4$?

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$$b - c = b - c + (a + b + c) = a + 2b = a - b$$

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$$b - c = b - c + (a + b + c) = a + 2b = a - b$$

This means that a SET corresponds to a line in $\mathbb{F}_3^4$.

Generalising SET

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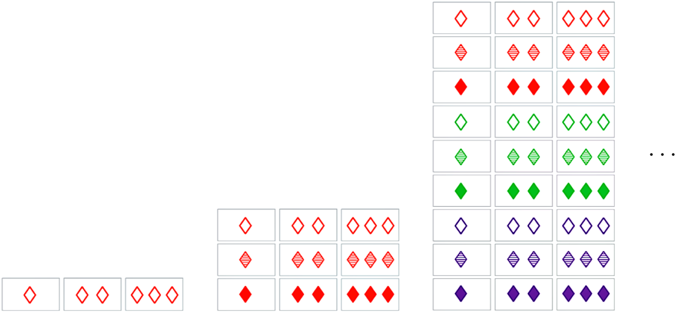

$d = 1$ $d = 2$ $d = 3$ ...

Generalising SET

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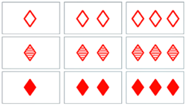
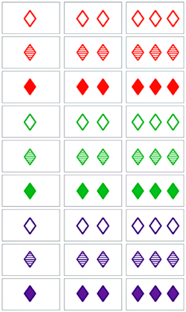
$d = 1$

$\mathbb{F}_3$



$d = 2$

$\mathbb{F}_3^2$



$d = 3$

$\mathbb{F}_3^3$

$\cdots$

$\cdots$

$\cdots$

d-Caps

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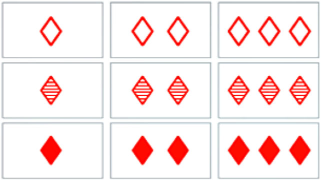
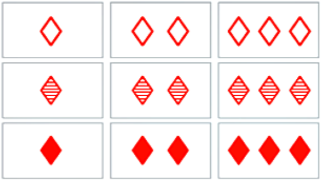
One interesting question here is what is the maximal size of a d-cap.
For $d = 4$, this corresponds to the question:

> In a standard game of SET, what is the maximal size of a collection of cards that contain no SETS?

Maximal 1-cap and 2-cap

	$d = 1$	$d = 2$
All cards		
Maximal d-cap		

Maximal 1-cap and 2-cap

	$d = 1$	$d = 2$
All cards		
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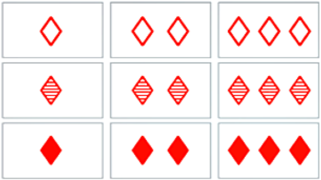
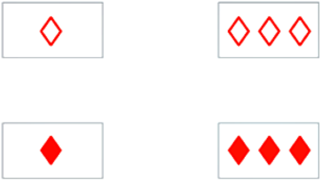
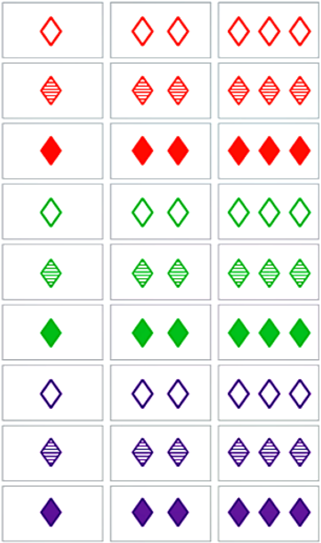
All cards		
Maximal d-cap		

Maximal 3-cap

How about for $d = 3$?

All cards

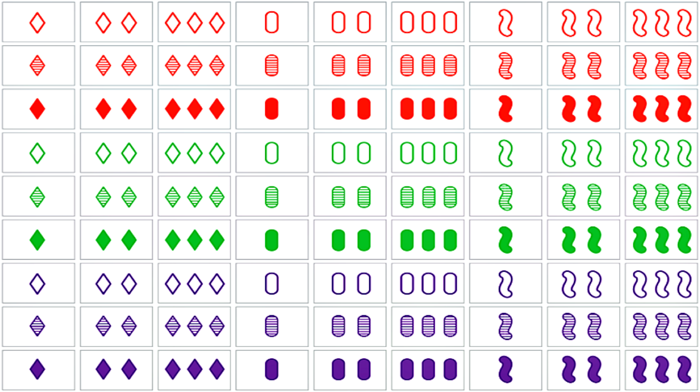


Maximal d-cap

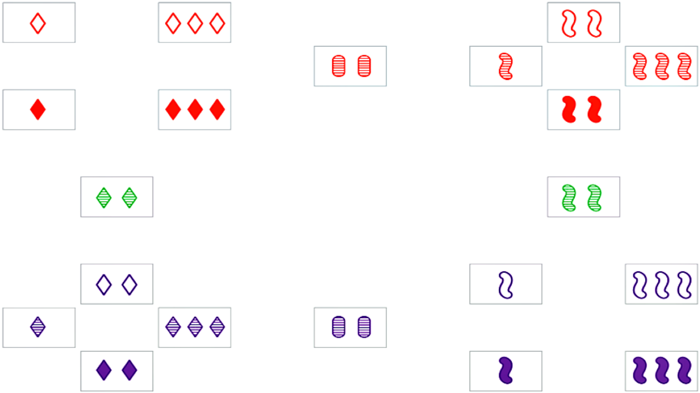
Maximal 3-cap

How about for $d = 3$?

Maximal 4-cap



Maximal 4-cap



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d	1	2	3	4	5	6	7
Size of Maximal d-cap	2	4	9	20			

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Size of Maximal d-cap	2	4	9	20	45	$112 \leq ? \leq 114$	?

The maximal size for $d = 5$ was found in 2006, using a novel approach to radically reduce the number of cases to check.
No further exact values are known.