Auslander-Reiten Quiver

Luke Kershaw

University of Bristol

l.kershaw@bristol.ac.uk

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Background

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The aim of the Auslander-Reiten quiver is to encode important information about a category of modules into a quiver.
We do this by only considering the "least complicated" modules and the "least complicated" homomorphisms between them.

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   - Here we introduce our notions of "least complicated" for modules and homomorphisms between them
2. Define quivers and discuss basic properties of them
3. Define the Auslander-Reiten quiver and discuss its basic properties
4. Discuss a basic example of the Auslander-Reiten quiver

Throughout we will assume K is an algebraically closed field.

Definition

A $\underline{K\text{-algebra}}$ is a ring A with an identity element (denoted by 1) such that A has a K-vector space structure compatible with the multiplication of the ring:

$$\lambda(ab) = (\lambda a)b = a(\lambda b) = (ab)\lambda$$

for all $\lambda \in K$ and $a, b \in A$.

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Definition

If A and B are K-algebras, then a ring homomorphism $f : A \to B$ *is called a K-algebra homomorphism if f is a K-linear map.*

Definition

Let A be a K-algebra. A right A-module is a pair $(M, \cdot)$, where M is a K-vector space and $\cdot : M \times A \to M; (m, a) \mapsto ma$ is a binary operation satisfying the following conditions:

(a) $(x + y)a = xa + ya$

(b) $x(a + b) = xa + xb$

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Let M and N be right A-modules. A K-linear map $h : M \to N$ is called an A-module homomorphism if $h(ma) = h(m)a$ for all $m \in M, a \in A$.

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A right A-module M is called indecomposable if M is nonzero and M has no direct sum decomposition $M \cong N \oplus L$, where L and N are nonzero A-modules.

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Lemma

If $f : X \to Y$ *is irreducible, then it is either a proper monomorphism or a proper epimorphism.*

Radical between Indecomposable Modules

Definition

If X and Y are modules in mod-A *then the* radical *is defined by:*

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If X and Y are indecomposable modules in mod-A, this definition simplifies to say that $\mathrm{rad}_A(X, Y)$ is the K-vector space of all noninvertible homomorphisms from X to Y. Thus for X indecomposable, $\mathrm{rad}_A(X, X)$ is just the radical of the local algebra $\mathrm{End}(X)$.

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If X and Y are modules in mod-A *then we define* $\text{rad}_A^2(X, Y)$ *to consist of all A-module homomorphisms of the form gf, where $f \in \text{rad}_A(X, Z)$ and $g \in \text{rad}_A(Z, Y)$ for some module Z in* mod-A.

Lemma

Let X, Y be indecomposable modules in mod-A. *A morphism $f : X \to Y$ is irreducible if and only if $f \in \operatorname{rad}_A(X, Y) \setminus \operatorname{rad}_A^2(X, Y)$.*

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Therefore the quotient:

$$\operatorname{Irr}(X, Y) := \operatorname{rad}_A(X, Y) / \operatorname{rad}_A^2(X, Y)$$

of the K-vector spaces $\operatorname{rad}_A(X, Y)$ and $\operatorname{rad}_A^2(X, Y)$ measures the number of irreducible morphisms from M to N.

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Definition

We call $\operatorname{Irr}(X, Y)$ the space of irreducible morphisms.

Quivers

Definition

A *quiver* $Q = (Q_0, Q_1, s, t)$ *is a quadruple consisting of two sets:* Q_0 *(the* *vertices**) and* Q_1 *(the* *arrows**), and two maps* $s, t : Q_1 \to Q_0$ *which associate to each arrow* $\alpha \in Q_1$ *its* *source* $s(\alpha) \in Q_0$ *and its* *target* $t(\alpha) \in Q_0$*.*

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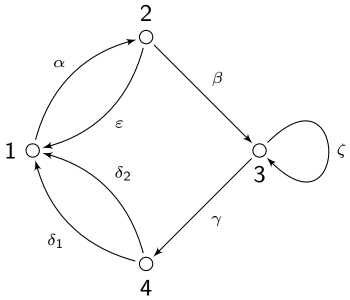
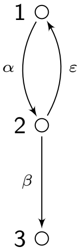
In other words, a quiver is a directed graph where loops and multiple edges are allowed. Some examples:





Neighbours in a Quiver

Definition

If there is a path in Q from a to b, then a is predecessor of b and b is a successor of a. In particular, if there exists an arrow $a \to b$, then a is a direct predecessor of b and b is a direct successor of a.
For $a \in Q_0$ we denote by a^- (resp. a^+) the set of all direct predecessors (resp. direct successors). The elements of $a^+ \cup a^-$ are called neighbours of a.

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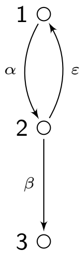
Example:



For $a = 1$, $a^+ = \{2\}$ and $a^- = \{2\}$.
For $a = 2$, $a^+ = \{1, 3\}$ and $a^- = \{1\}$.
For $a = 3$, $a^+ = \varnothing$ and $a^- = \{2\}$.

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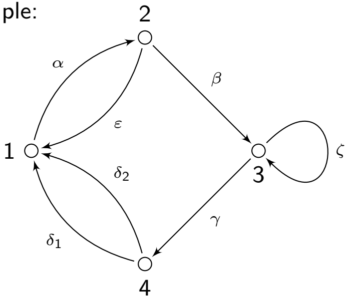

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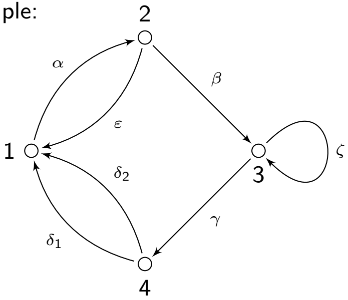

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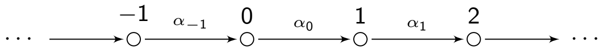
A quiver is locally finite if for each $x \in Q_0$, its neighbourhood $x^+ \cup x^-$ is finite.

Example of locally finite but infinite quiver:



Definition

To a category of modules mod-A*, we can associate its* *Auslander-Reiten quiver, denoted* $\Gamma = \Gamma(\text{mod-}A)$*, which is defined by:*

(a) *the vertices of* Γ *are the isomorphism classes* $[X]$ *of indecomposable modules* X *in* mod-A

(b) *for* $[M], [N]$ *points in* Γ *corresponding to the indecomposable modules* M, N *in* mod-A*, the arrows are in bijective correspondence with the vectors of a basis of the* K*-vector space* $\text{Irr}(M, N)$*.*

Immediate Properties of the Auslander-Reiten Quiver

It can be shown that each indecomposable module M has only finitely many indecomposable L such that there is an irreducible morphism $L \to M$. Similarly each indecomposable module M has only finitely many indecomposable N such that there is an irreducible morphism $M \to N$.
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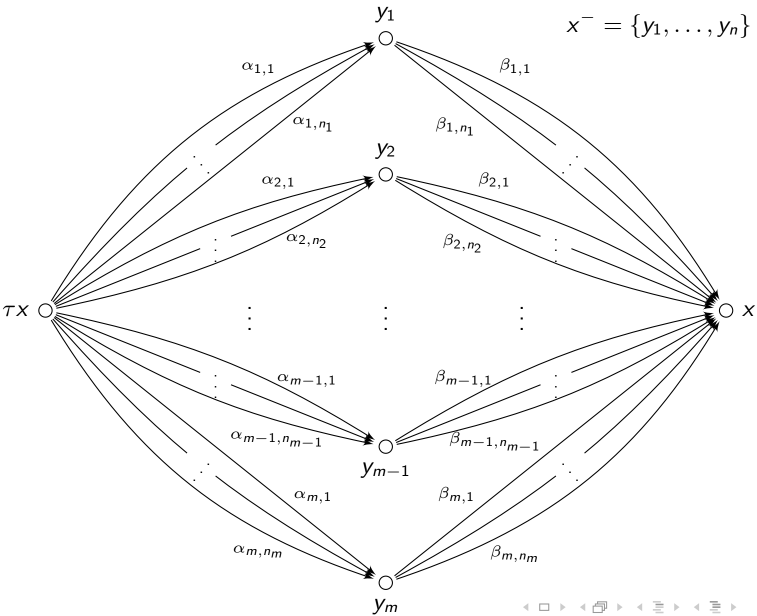
Clearly both of these are contradictions when M is finite dimensional, and thus there are no irreducible morphisms, $f : M \to M$. Hence $\Gamma(\text{mod-}A)$ has no loops.

The Auslander-Reiten quiver has a particular combinatorial property involving the neighbours of vertices.

Definition

Let Q be a locally finite quiver without loops and τ be a bijection whose domain and codomain are both subsets of Q_0. The pair (Q, τ) is called a translation quiver if for every $x \in Q_0$ such that τx exists, and every $y \in x^-$, the number of arrows from y to x equals the number of arrows from τx to y.

Translation Quiver Diagram



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The Auslander-Reiten quiver $\Gamma = \Gamma(\text{mod-}A)$ is a translation quiver when equipped with the quiver translation induced by the Auslander-Reiten translation ($D\,\mathrm{Tr}$) on modules, i.e. $\tau[M] = [D\,\mathrm{Tr}(M)]$.

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The Auslander-Reiten translation is an endofunctor defined as the composition of the standard duality functor $D = \text{Hom}_K(-, K) : \text{mod-}A \to \text{mod-}A^{\text{op}}$ (which is bijective and associates the projective modules to the injective modules) and the transposition functor $\text{Tr} : \text{mod-}A^{\text{op}} \to \text{mod-}A$ (which vanishes exactly on the projective modules).

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Thus we choose the domain of τ to be Γ_0', the set of those points in Γ_0 that correspond to a nonprojective indecomposable module. We also choose the codomain of τ to be Γ_0'', the set of those points in Γ_0 that correspond to a noninjective indecomposable module. With this choice of domain and codomain $\tau : \Gamma_0' \to \Gamma_0''$ is a bijection.

Path Algebras

Definition

Let $Q = (Q_0, Q_1, s, t)$ *be a quiver and* $a, b \in Q_0$. *A path of length* $l \geq 1$ *with source* a *and target* b *is a sequence*

$$(a|\alpha_1, \alpha_2, \ldots, \alpha_l|b)$$

where each $\alpha_k \in Q_1$ *and we have* $s(\alpha_1) = a, t(a_k) = s(a_{k+1}), t(a_l) = b$. *We also associate to each point* $a \in Q_0$ *a path of length* $l = 0$*, called the trivial path at* a *and denoted by* $\varepsilon_a = (a||a)$.

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Definition

Let Q be a quiver. The path algebra KQ of Q is the K-algebra whose underlying K-vector space has as its basis the set of all paths of length $l \geq 0$ in Q, and such that the product of two basis vectors $(a|\alpha_1, \ldots, \alpha_l|b)$ and $(c|\beta_1, \ldots, \beta_k|d)$ of KQ is defined by

$$(a|\alpha_1, \ldots, \alpha_l|b)(c|\beta_1, \ldots, \beta_k|d) = \delta_{bc}(a|\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_k|d)$$

where δ_{bc} denotes the Kronecker delta.

Example of Path Algebra

Consider the quiver:

$$Q: \quad \underset{\circ}{1} \xleftarrow{\beta} \underset{\circ}{2} \xleftarrow{\alpha} \underset{\circ}{3}$$

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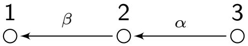

Then KQ is a 6-dimensional path algebra with basis $\mathcal{B} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \alpha, \beta, \alpha\beta\}$ and multiplication table:

$\cdot$	ε_1	ε_2	ε_3	α	β	$\alpha\beta$
ε_1	ε_1	0	0	0	0	0
ε_2	0	ε_2	0	0	β	0
ε_3	0	0	ε_3	α	0	$\alpha\beta$
α	0	α	0	0	$\alpha\beta$	0
β	β	0	0	0	0	0
$\alpha\beta$	$\alpha\beta$	0	0	0	0	0

Example of Auslander-Reiten Quiver

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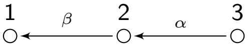

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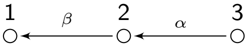

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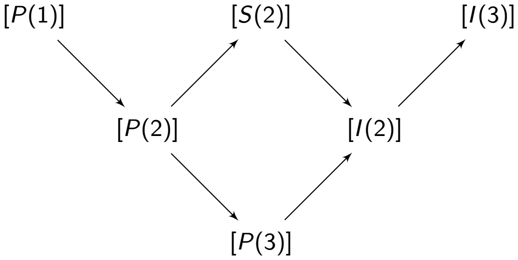
The irreducible morphisms between these classes are arranged as follows:



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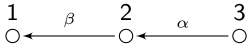

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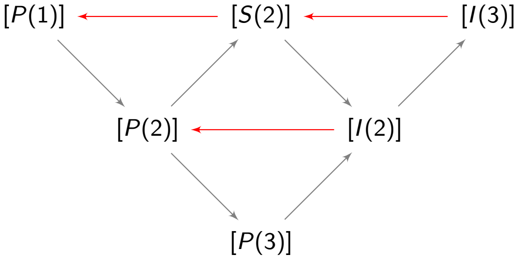

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Theorem

Assume that A is a "nice" finite dimensional K-algebra. If $\Gamma(\text{mod-}A)$ admits a connected component $\mathcal{C}$ whose modules are of bounded length, then $\mathcal{C}$ is finite and $\mathcal{C} = \Gamma(\text{mod-}A)$. In particular, A has finitely many isomorphism classes of indecomposable modules (i.e. A is representation finite).

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Corollary

Let A be a representation-finite algebra. Any nonzero nonisomorphism between indecomposable modules in $\text{mod-}A$ is a sum of compositions of irreducible morphisms.

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Assume that A is a "nice" finite dimensional K-algebra. If $\Gamma(\text{mod-}A)$ admits a connected component $\mathcal{C}$ whose modules are of bounded length, then $\mathcal{C}$ is finite and $\mathcal{C} = \Gamma(\text{mod-}A)$. In particular, A has finitely many isomorphism classes of indecomposable modules (i.e. A is representation finite).

Corollary

Let A be a representation-finite algebra. Any nonzero nonisomorphism between indecomposable modules in $\text{mod-}A$ is a sum of compositions of irreducible morphisms.

The proofs of both of these results are very technical, so I will not go into them here.